Lecture 15

Chapter 3.1 + 3.2 - Linear Independence

$\text{span}\left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix} \right\} = \mathbb{R}^2$

redundant

$$\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}$$

So e.g. $5\begin{pmatrix}1\\0\end{pmatrix} + (-2)\begin{pmatrix}0\\1\end{pmatrix} + 3\begin{pmatrix}1\\1\end{pmatrix}$

$= 5\begin{pmatrix}1\\0\end{pmatrix} + (-2)\begin{pmatrix}0\\1\end{pmatrix} + 3\left[\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}\right]$

$= 8\begin{pmatrix}1\\0\end{pmatrix} + (1)\begin{pmatrix}0\\1\end{pmatrix}$

Alternatively: we have a "linear relation" $\begin{pmatrix}1\\1\end{pmatrix} - \begin{pmatrix}1\\0\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$

$$\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} - \begin{pmatrix}1\\0\end{pmatrix}$$

so $\text{span}\left\{ \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix} \right\} = \text{span}\left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\} = \text{span}\left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix} \right\}$

Def - A <u>linear relation</u> among vectors $\vec{v}_1, \dots, \vec{v}_k \in \mathbb{R}^n$ is a list of scalers $c_1, \dots, c_k \in \mathbb{R}$ such that $c_1\vec{v}_1 + \dots + c_k\vec{v}_k = \vec{0}$

Rmk - Always have the <u>trivial</u> relation $0\vec{v}_1 + \dots + 0\vec{v}_k = \vec{0}$

Rmk - When $\exists$ a non-trivial relation $c_1\vec{v}_1 + \dots c_k\vec{v}_k = \vec{0}$ we can solve for one of the vecs in terms of the others.

$\boxed{\text{Ex}}$ if $c \neq 0$ then $\vec{v}_1 = \frac{-c_2}{c_1}\vec{v}_2 + \dots + \frac{-c_k}{c_1}\vec{v}_k$

$\Rightarrow \text{span}\{\vec{v}_1, \dots, \vec{v}_k\} = \text{span}\{\vec{v}_2, \dots, \vec{v}_k\}$

Def - Vectors $\vec{v}_1, \dots, \vec{v}_K \in \mathbb{R}^n$ are linearly independent if $c_1\vec{v}_1 + \dots + c_K\vec{v}_K = \vec{0}$ only when $c_1 = \dots = c_K = 0$.

Alternative - $\vec{v}_1, \dots, \vec{v}_K \in \mathbb{R}^n$ are LI if $\exists$ no non-trivial linear relations among them.

Cases (K is number of vectors):

$K=1$ $\quad \vec{v} \in \mathbb{R}^n$ is LI i.f.f. $\vec{v} \neq \vec{0}$.

$K=2$ $\quad \vec{v}, \vec{w} \in \mathbb{R}^n$ are LI i.f.f. neither is a multiple of the other.

$K=3$:

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = \vec{0}$$

i.e. $A = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3]$, $\vec{x} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$

$$A\vec{x} = \vec{0}$$

So row reduce $[A \mid \vec{0}]$

- If we end with free variables, the vecs $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are dependent.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} x_3$$

Set $x_3 = 3 \Rightarrow \vec{x}_1 = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 9 \\ -3 \\ 3 \end{pmatrix}$ gives us $9\vec{v}_1 - 3\vec{v}_2 + 3\vec{v}_1 = \vec{0}$

$$\vec{v}_2 = 3\vec{v}_1 + \vec{v}_3$$

$$\Rightarrow \text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \text{span}\{\vec{v}_2, \vec{v}_3\}$$

- The standard choice of vectors to eliminate are the ones corresponding to free variables.

In the Big eg from last time:

$$A\vec{x}=\vec{0} \Rightarrow \vec{x} = \vec{x}_2\begin{pmatrix}-4\\1\\0\\0\\0\end{pmatrix} + \vec{x}_4\begin{pmatrix}-2\\0\\1\\1\\0\end{pmatrix} + \vec{x}_5\begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix}$$

$$A = \begin{bmatrix}\vec{v}_1 \dots \vec{v}_K\end{bmatrix}$$

$\Rightarrow$ non trivial linear relations

$\Rightarrow$ So the columns $\vec{v}_1 \dots \vec{v}_5$ of A are dependent.

Claim further: $\vec{v}_1$, $\vec{v}_3$ are linearly independent and $\operatorname{span}\{\vec{v}_1, \vec{v}_3\} = \operatorname{span}\{\vec{v}_1, \dots, \vec{v}_5\}$

Why does this work?

$\boxed{\text{Eg}}$ Why can $\vec{v}_2$ go?

ANS - Set $x_2 = 1$, $x_4 = x_5 = 0$

$$\Rightarrow \vec{x} = \begin{bmatrix}-4\\1\\0\\0\\0\end{bmatrix} \Longrightarrow -4\vec{v}_1 + \vec{v}_2 = \vec{0} \Rightarrow \vec{v}_2 = 4\vec{v}_1$$

i.e. I can write 2^{nd} col of A as a linear combination of the pivot columns.

Likewise: setting $x_4 = 1$, $x_2 = x_5 = 0$

gives $\vec{x} = \begin{pmatrix}-2\\0\\1\\1\\0\end{pmatrix}$ $\quad -2\vec{v}_1 + \vec{v}_3 + \vec{v}_4 = \vec{0}$

$\vec{v}_4 = 2\vec{v}_1 - \vec{v}_3$

$\Rightarrow \vec{v}_4$ can go